Lower Bounds on the PTF Weight of ODD-MAXBIT Function

Kazuyuki Amano*

October 5, 2022

Abstract

We show that every polynomial threshold function that sign-represents the $\mathsf{ODD\text{-}MAXBIT}_n$ function has total absolute weight $2^{\Omega(n^{1/3})}$. The bound is tight up to a logarithmic factor in the exponent.

1 Introduction and Results

In this note, we investigate the polynomial threshold representation of Boolean functions. This representation has been extensively studied in areas such as complexity theory and learning theory (see e.g., [1, 3, 4, 5]).

We say that a multivariate polynomial $p : \{0,1\}^n \to \mathbb{Z}$ with integer coefficients *sign-represents* a Boolean function $f : \{0,1\}^n \to \{-1,1\}$ if, for every $\boldsymbol{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$, $f(\boldsymbol{x}) = \mathsf{sign}(p(\boldsymbol{x}))$, i.e.,

$$\begin{aligned} f(x_1,\ldots,x_n) = 1 \quad &\Rightarrow \quad p(x_1,\ldots,x_n) \geq 1, \\ f(x_1,\ldots,x_n) = -1 \quad &\Rightarrow \quad p(x_1,\ldots,x_n) \leq -1. \end{aligned}$$

We also say that such a polynomial is a PTF (polynomial threshold function) for f.

Since $x^2 = x$ for $x \in \{0,1\}$, we only need to consider a multilinear polynomial. The *degree* of a polynomial p is the maximum number of variables appearing in any term in p.

For a polynomial p, the *weight* of p, denoted by $W(p)$, is defined as the sum of absolute coefficients in p. For a Boolean function f, the *PTF weight of f* is defined as the smallest $W(p)$ over all PTFs p for f.

*Gunma Univerisity, Japan (amano@gunma-u.ac.jp)

Definition 1 *The ODD-MAXBIT$_n$ function is an n-variable Boolean function that takes $(x_1, \ldots, x_n) \in \{0,1\}^n$ as an input and outputs $(-1)^{i-1}$ if i is the largest index $i \in \{1, \ldots, n\}$ such that $x_i = 1$. If an input is 0^n, then it outputs -1. In other words, ODD-MAXBIT$_n$ outputs 1 iff the rightmost bit of an input that is set to 1 is in an odd bit position.* □

Note that ODD-MAXBIT$_n$ can be represented by a decision list of length n. The aim of this note is to show the following lower bound on the PTF weight of ODD-MAXBIT$_n$.

Theorem 1 *The PTF weight of ODD-MAXBIT$_n$ is $2^{\Omega(n^{1/3})}$.*

Note that a PTF is usually considered over the domain $\{0,1\}^n$ or $\{-1,1\}^n$ and that our result is valid only for a PTF over $\{0,1\}^n$.

By the following upper bound by Podolskii and Proskurin [6] (which improves a slightly larger bound by Kilivan and Servedio [3]), our lower bound is tight up to a logarithmic factor in the exponent.

Theorem 2 (Podolskii and Proskurin [6]) *Let f be a Boolean function that can be represented by a decision list of length n. Then f can be represented by a PTF of degree $O(n^{1/3})$ and weight $2^{O(n^{1/3} \log n)}$.*

Prior to this work, an exponential lower bound on the PTF weight of ODD-MAXBIT$_n$ has been known only for degree-d polynomial. Beigel [2] showed that every degree-d PTF for ODD-MAXBIT$_n$ has weight $2^{\Omega(n/d^2)}$, and Servedio, Tan and Thaler [5] showed that every degree-d PTF for ODD-MAXBIT$_n$ has weight $2^{\Omega(\sqrt{n/d})}$.

2 Proof of Theorem 1

The proof of Theorem 1 relies on the random restriction and the self-reducibility of the ODD-MAXBIT$_n$ function.

We will use the following lower bound on the weight of a degree-d PTF for ODD-MAXBIT$_n$ due to Beigel [2].

Theorem 3 (Beigel [2]) *Suppose that p is a PTF for ODD-MAXBIT$_n$. If the degree of p is at most d, then $W(p) = 2^{\Omega(n/d^2)}$.* □

Proof of Theorem 1. Suppose without loss of generality that n is even. Let p be any PTF for $\mathsf{ODD\text{-}MAXBIT}_n$. We will show that $W(p) = 2^{\Omega(n^{1/3})}$.

Let $\ell := n^{1/3}$. Let $W^+(p)$ denote the sum of absolute coefficients of all terms in p whose degree is at least ℓ.

If $W^+(p) \geq 2^{0.1n^{1/3}}$, then we are done. We now assume that $W^+(p) < 2^{0.1n^{1/3}}$.

Let U be a uniform distribution over all partial assignments to the input variables such that $0.1n$ variables are set to 0 and remaining $0.9n$ variables are unassigned.

We say that a term t is *hit* by a partial assignment $\rho \in \{0, *\}^n$ if t contains a variable that is assigned 0 in ρ. Obviously, a term is vanished under ρ when it is hit by ρ. Following the standard notation, we write a function f or a polynomial p under a partial assignment ρ as $f|_\rho$ or $p|_\rho$, respectively.

For every term t in p whose degree ℓ' is at least ℓ, we see that

$$\begin{aligned}
&\Pr_{\rho\sim U}[t \text{ is not hit by } \rho] \\
&= \frac{\binom{n-\ell'}{0.1n}}{\binom{n}{0.1n}} \leq \frac{\binom{n-\ell}{0.1n}}{\binom{n}{0.1n}} \\
&= \frac{n-\ell}{n} \cdot \frac{n-\ell-1}{n-1} \cdot \cdots \cdot \frac{0.9n-\ell+1}{0.9n+1} \\
&\leq \left(\frac{n-\ell}{n}\right)^{0.1n} \\
&= \left(1-\frac{\ell}{n}\right)^{0.1n} \leq e^{-0.1\ell}.
\end{aligned}$$

By the linearity of expectation, we have

$$\begin{aligned}
\mathbf{E}_{\rho\sim U}[W^+(p|_\rho)] &\leq W^+(p) \cdot e^{-0.1\ell} \\
&< 2^{0.1n^{1/3}} \cdot e^{-0.1n^{1/3}} = o(1),
\end{aligned}$$

where the symbol $\mathbf{E}$ stands for expectation. This implies that there exists a partial assignment $\rho' \in \{0, *\}^n$ such that ρ' has $0.1n$ 0's and $W^+(p|_{\rho'}) = 0$.

We now introduce another partial assignment $\tilde{\rho} \in \{0, *\}^n$ obtained from ρ' by additionally assigning the value 0 to some other variables. We will ensure that $\tilde{\rho}(x_{2k-1}, x_{2k}) = (0, 0)$ or $(*, *)$, for every $1 \leq k \leq n/2$ and that the number of variables that are assigned 0 is $0.2n$.

Precisely, the partial assignment $\tilde{\rho}$ can be obtained from ρ' by additionally assigning $x_{2k} := 0$ if x_{2k-1} is assigned 0 in ρ' and $x_{2k-1} := 0$ if x_{2k} is assigned 0 in ρ', for each

$1 \leq k \leq n/2$. If necessary, we also assign $(x_{2k-1}, x_{2k}) := (0, 0)$ for an appropriate number of an arbitrarily chosen k.

A key to the proof is the self-reducibility of $\mathsf{ODD\text{-}MAXBIT}_n$, i.e., $\mathsf{ODD\text{-}MAXBIT}_n|_{\tilde{\rho}}$ is equivalent to $\mathsf{ODD\text{-}MAXBIT}_{n-0.2n} \equiv \mathsf{ODD\text{-}MAXBIT}_{0.8n}$ under a suitable renaming of input variables. This is clear by considering a decision list representing $\mathsf{ODD\text{-}MAXBIT}_n$.

Since $W^+(p|_{\tilde{\rho}}) = 0$, $p|_{\tilde{\rho}}$ is a degree-ℓ ($=n^{1/3}$) PTF for $\mathsf{ODD\text{-}MAXBIT}_{0.8n}$. We can now apply Theorem 3 to obtain $W(p) \geq W(p|_{\tilde{\rho}}) = 2^{\Omega(n^{1/3})}$. This completes the proof of Theorem 1. □

Acknowledgement

This work was supported in part by JSPS Kakenhi No. JP21K19758, JP18K11152 and JP18H04090.

References

[1] K. Amano and S. Tate, On XOR lemmas for the weight of polynomial threshold functions, Inf. Comput., 269, article 104439 (2019)

[2] R. Beigel, Perceptrons, PP, and the polynomial hierarchy, Comput. Complex., 4, 339–349 (1994)

[3] A.R. Klivans and R.A. Servedio, Learning DNF in time $2^{\tilde{O}(n^{1/3})}$. J. Comput. Sys. Sci., 68(2), 303–318 (2004)

[4] R. O'Donnell and R.A. Servedio, Extremal properties of polynomial threshold functions, J. Comput. Sys. Sci., 74(3), 298–312 (2008)

[5] R.A. Servedio, L. Tan and J. Thaler, Attribute-efficient learning and weight-degree tradeoffs for polynomial threshold functions, Proc. of COLT 2012, 14.1–14.19 (2012)

[6] V. Podolskii and N.V. Proskurin, Polynomial threshold functions for decision lists, `ArXiv:2207.09371` (2022)