# On Optimal Merging Networks

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**Abstract.** We prove that Batcher's odd-even $(m, n)$-merging networks are *exactly* optimal for $(m, n) = (3, 4k + 2)$ and $(4, 4k + 2)$ for $k \geq 0$ in terms of the number of comparators used. For other cases where $m \leq 4$, the optimality of Batcher's $(m, n)$-merging networks has been proved. So we can conclude that Batcher's odd-even merge yields optimal $(m, n)$-merging networks for every $m \leq 4$ and for every $n$. The crucial part of the proof is characterizing the structure of optimal $(2, n)$-merging networks.

## 1 Introduction

A *comparator network*, which consists of *comparators*, has been widely investigated as a model of oblivious comparison-based algorithm. A comparator is a 2-input 2-output gate, where one output computes the minimum of the two inputs and the other output computes the maximum (see Fig. 1).

An $(m, n)$-merging network is a comparator network that merges $m$ elements $x_1 \leq x_2 \leq \cdots \leq x_m$ with $n$ elements $y_1 \leq y_2 \leq \cdots \leq y_n$ to form the sorted sequence $z_1 \leq z_2 \leq \cdots \leq z_{m+n}$. A merging network is usually drawn as in Fig. 2.

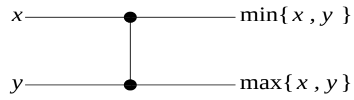


**Fig. 1.** A comparator.

Batcher[2] proposed *odd-even* merge, which gives an $(m, n)$-merging network with the least number of comparators used known up to the present. Let $C(m, n)$ be the number of comparators used in the odd-even merge for $m$ and $n$. The function $C(m, n)$ is given(see [4, p.224]) by

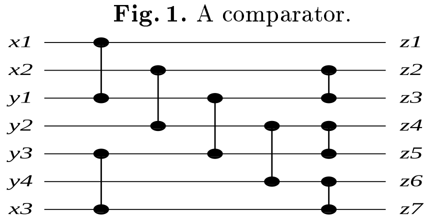


**Fig. 2.** A $(3, 4)$-merging network.

$$C(m,n) = \begin{cases} mn, & \text{if } mn \leq 1; \\ C(\lceil m/2 \rceil, \lceil n/2 \rceil) + C(\lfloor m/2 \rfloor, \lfloor n/2 \rfloor) + \lfloor (m+n-1)/2 \rfloor, & \text{if } mn > 1. \end{cases}$$

Let $M(m, n)$ denote the minimum number of comparators in an $(m, n)$-merging network. At present no merging networks have been discovered that are superior to the odd-even merge, and the problem of proving or disproving $M(m, n) = C(m, n)$ for every $m$ and $n$ remains unsolved over 30 years (see [8] or [4, Exercise 43, p. 241]). We call an $(m, n)$-merging network that uses $M(m, n)$ comparators by *optimal* merging network.

There has been considerable effort to determine the exact values of $M(m, n)$. It is clear that $M(1, n) = C(1, n) = n$. Yao and Yao[8] proved that $M(2, n) = C(2, n) =$

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$\lceil 3n/2 \rceil$. Aigner and Schwarzkopf[1] showed that $M(3,n) = C(3,n) = \lceil (7n+3)/4 \rceil$ for $n \equiv 0,1,3 \bmod 4$. In the same paper, they also stated that $M(3,n) = C(3,n)$ for $n \equiv 2 \bmod 4$ but without a detailed proof. Recently, Iwata[3] proved that

$$M(m_1+m_2,n) \geq \lceil M(m_1,n) + M(m_2,n) + m_1 + m_2 + n - 2)/2 \rceil, \qquad (1)$$

for every $m_1, m_2 \geq 1$ and for every $n \geq 1$. As a corollary of this inequality, they showed that $M(m,n) = C(m,n)$ for $m = 3,4$ and for $n \equiv 0,1,3 \bmod 4$. Unfortunately, for $n \equiv 2 \bmod 4$, the best lower bounds obtained by Eq. (1) is $M(m,n) \geq C(m,n) - 1$ for $m = 3,4$. Somewhat interestingly, the case $n \equiv 2 \bmod 4$ seems hardest to determine the exact values of $M(m,n)$ for $m = 3,4$. The exact values of $M(n,n)$ for $n \leq 9$ have been known[1, 7], and the asymptotic behavior of $M(m,n)$ has been also investigated in, e.g., [3, 5, 6, 8].

The main result of the paper is to show that Batcher's odd-even merge yields optimal $(m,n)$-merging networks for every $m \leq 4$ and for every $n$. This is achieved by proving $M(3,4k+2) = C(3,4k+2)$ (in Section 3), and $M(4,4k+2) = C(4,4k+2)$ (in Section 4). The crucial part of the proofs of the above results is characterizing the structure of optimal $(2,n)$-merging networks(Theorem 1), which we describe in Section 2. In addition, our arguments can also be used to determine $M(m,n)$ for some $m > 4$, such as $(m,n) = (5,8k+6)$. This will be discussed in Section 5.

In what follows, we call a horizontal line of input element $x_i$ ($y_j$) of an $(m,n)$-merging network *line* $x_i$ (*line* $y_j$) for $1 \leq i \leq m$ ($1 \leq j \leq n$, respectively). The line $x_i$ is always placed above the line $x_{i+1}$ for every $1 \leq i < m$, and similarly for $y_j$. We can place lines $x_1, x_2, \ldots, x_m$ of an $(m,n)$-merging network interspersed within lines $y_1, y_2, \ldots, y_n$ in arbitrarily order since, for any two differently interspersed input lines, there is a transformation from an $(m,n)$-merging network with interspersed input lines into a network with another intersperse lines that preserves the number of comparators used [4, Exercise 16, p.238]. A comparator connecting a line $i$ to a line $j$ is denoted by $[i:j]$. A comparator is said to be *of the form* $[i:*]$ if it is a comparator $[i:j]$ for some $j$. Comparator of the form $[*:j]$ is defined similarly. A *subnetwork* of a merging network $N$ consists of a set $K$ of adjacent lines in $N$ with the sets of all comparators that connect two lines in $K$. We sometimes identify a merging network with the set of comparators contained in the network. For a set $A$, $|A|$ denotes the cardinality of $A$.

## 2 Structure of Optimal $(2,n)$-Merging Networks

In this section, we characterize the structure of optimal $(2,n)$-merging networks for even $n$. This is the key to the proofs of the lower bounds for $M(m,n)$ for $m = 3,4$ and we believe that it is interesting in its own right.

Put $n = 2k$. Let $A$ be an optimal $(2,n)$-merging network consisting of input lines $x_1, x_2, y_1, y_2, \ldots, y_n$ from top to bottom. The network $A$ contains $M(2,n) = \lceil 3n/2 \rceil = 3k$ comparators. Consider the behavior of the network $A$ for two input sequences $\boldsymbol{a}_0 = \langle 0, n+1, 1, 2, \ldots, n \rangle$ and $\boldsymbol{a}_1 = \langle n+1, n+2, 1, 2 \ldots, n \rangle$. Note that the network $A$ sorts $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$. We also note that if a network sorts $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$, then it is a $(2,n)$-merging network (this can be proved by using the zero-one principle[4, p. 224] or see [1, Lemma 1]). Now we divide the set of comparators $A$ into three subsets $A_0$, $A_1$ and $A_{01}$ as follows:

$A_0$ : the set of comparators that change the content of $\boldsymbol{a}_0$, but not of $\boldsymbol{a}_1$,
$A_1$ : the set of comparators that change the content of $\boldsymbol{a}_1$, but not of $\boldsymbol{a}_0$,
$A_{01}$ : the set of comparators that change the contents of $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$.

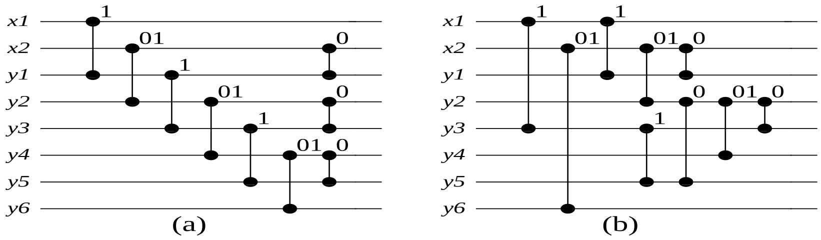


**Fig. 3.** (a) Batcher's $(2,6)$-merging network; (b) a $(2,6)$-merging network that uses the same number of comparators as Batcher's network. A comparator labeled "0" ("1" and "01") is a member of $A_0$ ("$A_1$" and "$A_{01}$", respectively).

Note that $A_0$, $A_1$ and $A_{01}$ are mutually disjoint, and $A_0 \cup A_1 \cup A_{01} = A$ since $A$ is optimal. In both networks shown in Fig. 3, the three sets $A_0$, $A_1$ and $A_{01}$ have the same size (,i.e., $|A_0| = |A_1| = |A_{01}| = 3$). Moreover, we can notice that every comparator in $A_0$ has odd "length" and every comparator in $A_1 \cup A_{01}$ has even "length". Formally, the *length* of a comparator $c = [i : j]$ is defined to be $(j - i)$ where we number the input lines $0, 1, 2, \ldots,$ from top to bottom. Note that in the network $A$, a comparator $[y_i : y_j]$ has length $(j - i)$ and $[x_i : y_j]$ has length $j + (2 - i)$. In the following, we prove that these conditions are satisfied by *any* optimal $(2, 2k)$-merging network[1]. The statements (iv) and (v) of Theorem 1 are crucial to the proofs of all our lower bounds.

**Theorem 1.** *For every $k \geq 1$ and for every optimal $(2, 2k)$-merging network $A$, which consists of lines $x_1, x_2, y_1, \ldots, y_{2k}$, the following are true:*

*(i)* $|A_0| = |A_1| = |A_{01}| = k$.
*(ii) For every $1 \leq i \leq 2k$, there is a unique comparator in $A_0 \cup A_{01}$ which is of the form $[* : y_i]$.*
*(iii) For every $1 \leq i \leq 2k$, there is a unique comparator in $A_1 \cup A_{01}$ which is of the form $[* : y_i]$.*
*(iv) Every comparator in $A_{01} \cup A_1$ has even length.*
*(v) Every comparator in $A_0$ has odd length.*

*Proof (of (i), (ii) and (iii), Theorem 1).* Recall that $n = 2k$. Let $A$ be an optimal $(2, 2k)$-merging network. Obviously, a network consisting of the set of comparators $A_0 \cup A_{01}$ sorts $\boldsymbol{a}_0$. For any network sorting $\boldsymbol{a}_0$, there are precisely $n$ comparators that change the content of $\boldsymbol{a}_0$ and there is a unique comparator which is of the form $[* : y_j]$ for every $1 \leq j \leq n$. (This was observed by Aigner and Schwarzkopf[1, Section 3], and a similar observation was made by Yao and Yao[8, Lemma 1].) This implies (ii) of Theorem 1, and $|A_0| + |A_{01}| = n = 2k$. Hence we have

$$|A_1| = 3k - 2k = k. \tag{2}$$

Let $(\boldsymbol{a}_0)_j$ $((\boldsymbol{a}_1)_j)$ be the content of the line $j$ after the $h$-th comparison in $A$ when $\boldsymbol{a}_0$ ($\boldsymbol{a}_1$, respectively) is given to $A$ as input. Let us say that a line $j$ is *split* after the $h$-th

[1] Note that it seems there are exponentially many optimal merging networks. We did an exhausting search and found that there are 9 optimal $(2,4)$-merging networks and 140 optimal $(2,6)$-merging networks, while an optimal $(2,2)$-merging network is unique.

comparison if $(\boldsymbol{a}_0)_j \neq (\boldsymbol{a}_1)_j$ holds. We define $d_h$ to be the number of split lines after the $h$-th comparison. Thus, at the input $d_0 = 2$, and at the end $d_{3k} = n + 2$, and hence $d_{3k} - d_0 = n$. Since every comparator raises the value of $d_h$ at most 1 and a comparator in $A_{01}$ does not change the value of $d_h$, we have $|A_0| + |A_1| \geq n = 2k$. By Eq. (2), we have $|A_0| \geq k$.

Here we show that, for every comparator $c$ of the form $[* : y_j]$ in $A_0$, there is another comparator of the form $[* : y_j]$ in $A_1$ that lies to the left of $c$. If this is proven, then $k = |A_1| \geq |A_0| \geq k$, and hence $k = |A_1| = |A_0| = |A_{01}|$ (that is, (i) of Theorem 1.). Moreover, this and (i) and (ii) of Theorem 1 imply (iii) of the theorem.

Suppose that a comparator $c = [i : y_j]$ is in $A_0$. It is easy to check that we must have $(\boldsymbol{a}_0)_{y_j} < (\boldsymbol{a}_0)_i \leq (\boldsymbol{a}_1)_i < (\boldsymbol{a}_1)_{y_j}$ at the input of $c$. Thus, the line $y_j$ must be split before encountering the comparator $c$. Since the initial content of the line $y_j$ is $(j, j)$, the line $y_j$ must have been part of a previous comparator. Since the content of $y_j$ is greater than or equal to $j$ at any stage, the first comparator changing the content $(j, j)$ is of the form $[* : y_j]$. Moreover this comparator must be in the set $A_1$ since $c$ is in $A_0$ and (ii) of Theorem 1. This completes the proof of (iii) of the theorem. □

*Proof (of (iv), Theorem 1).* Let us focus on the content of $\boldsymbol{a}_1$ in the network $A$. We number the lines $1, 2, \ldots, n+2$ from top to bottom. For any stage in $A$, the *differential* of line $j$, denoted by $e_j$, is defined to be $(\boldsymbol{a}_1)_j - j$ and the *differential sequence* is defined to be $\langle e_1, e_2, \ldots, e_{n+2} \rangle$. Note that the differential sequence is $\langle n, n, -2, \ldots, -2 \rangle$ at the input and is $\langle 0, 0, \ldots, 0 \rangle$ at the output.

If there is a comparator of odd length in $A_{01} \cup A_1$, then there must be a line whose differential is an odd. (This is proved by focusing on the outputs of the first such comparator.) So to prove (iv), it is sufficient to show that there are no lines whose differential $d$ is odd in the network $A$. We shall prove this by induction on $t$ where $d = 2t + 1$.

Let $c = [i : j] \in A_{01} \cup A_1$ be a comparator whose length is $l$. Suppose that $(e_i, e_j)$ is equal to $(\alpha, \beta)$ at the input of $c$. By (iii) of Theorem 1 and the first comparator changing the content of line $j$ is of the form $[* : y_j]$, we have $\beta = -2$. Since the comparator $c$ swaps the two inputs, we have $i + \alpha > -2 + j$, and this implies $\alpha + 2 > j - i = l \geq 1$. Hence $\alpha \geq 0$. So every comparator in $A_{01} \cup A_1$ decreases the number of negative entries in a differential sequence by at most 1. Because it contains precisely $n = |A_{01} \cup A_1|$ negative entries at the input, every comparator in $A_{01} \cup A_1$ must decrease the number of negative entries in a sequence by 1. The differential $(e_i, e_j)$ is changed from $(\alpha, -2)$ to $(-2 + l, \alpha - l)$ by the comparator $c$. Hence, for every comparator $c = [i : j] \in A_{01} \cup A_1$, the differential $(e_i, e_j) = (\alpha, -2)$ before $c$ must satisfy $\alpha \geq l \geq 2$. This proves the base case ($t = 0$).

For the induction step, suppose that $(e_i, e_j) = (2t + 1, -2)$ at the input of some comparator $c = [i : j] \in A_{01} \cup A_1$ whose length is $l$. The differentials of the outputs of $c$ are $-2 + l$ and $2t + 1 - l$. Since $2 \leq l \leq 2t + 1$, one of them is odd and is smaller than or equal to $2(t - 1) + 1$. This contradicts the induction hypothesis. (End of the proof of (iv) of Theorem 1.) □

*Proof (of (v), Theorem 1).* To prove (v) of Theorem 1, we will show that the number of comparators having odd length is greater than or equal to that having even length for any network of size $n$ that sorts $\boldsymbol{a}_0$. This implies (v) of Theorem 1 because (a) $A_0 \cup A_{01}$ sorts $\boldsymbol{a}_0$, (b) $|A_0| = |A_{01}| = n/2$ (Theorem 1, (i)) and (c) every comparator in $A_{01}$ has even length(Theorem 1, (iv)).

To show this, we analyze an arbitrary network with $n$ comparisons sorting $\boldsymbol{a} = \langle n+1, 1, 2, \ldots, n\rangle$, which consists of the lines $1, 2, \ldots, n+1$. Aigner and Schwarzkopf[1] observed that, for every such network, the content $a_1, \ldots, a_{n+1}$ after the $r$-th comparison can be described as follows: The set of lines $\{1, 2, \ldots, n+1\}$ is uniquely divided into groups $F_1|F_2|\cdots|F_{r+1}$ from top to bottom such that

(i) $i \in F_k, j \in F_l, k < l \Rightarrow a_i < a_j$.
(ii) if $F_k = \{f_k, f_k+1, \ldots, f_{k+1}-1\}$ then $a_{f_k+1} < \cdots < a_{f_{k+1}-1} < a_{f_k}$.

In other words, within each group the top element is the largest, with the others appearing in their natural order. A comparator $[i:j]$ changes the content of $\boldsymbol{a}$ if and only if $i$ and $j$ are in a certain group $F_k$ with $i = f_k < j \leq f_{k+1}-1$. By this comparator the set $F_k$ splits into two groups $F_k^u = \{f_k, f_k+1, \ldots, j-1\}$ and $F_k^l = \{j, j+1, \ldots, f_{k+1}-1\}$ satisfying (i) and (ii) when we replace $F_k$ by $F_k^u$ and $F_k^l$.

For a network $M$, let $o(M)$ ($e(M)$) be the number of comparators having odd length (even length, respectively) in $M$. Let $T(k)$ be the minimum value of $o(A_k) - e(A_k)$ where $A_k$ ranges over all networks with input lines $1, 2, \ldots, k+1$ consists of $k$ comparators that sorts a sequence $\langle a_1, a_2, \ldots, a_{k+1}\rangle$ with $a_2 < a_3 < \cdots < a_{k+1} < a_1$. Then $T(k)$ can be expressed recursively as follows:

$$T(k) = \min_{1 \leq i \leq k} \{(-1)^{i-1} + T(i-1) + T(k-i)\}. \tag{3}$$

We are going to show $T(k) = 0$ for every even $k$ and $T(k) = 1$ for every odd $k$ by the induction on $k$. The base cases $T(0) = 0$ and $T(1) = 1$ are obvious.

If $k$ is even, then by Eq. (3) and the induction hypothesis we have

$$\begin{aligned} T(k) = \min \Bigg\{ & -1 + \min_{k_1 : k_1 < k, k_1 \text{is odd}} \{T(k_1)\} + \min_{k_2 : k_2 < k, k_2 \text{is even}} \{T(k_2)\}, \\ & 1 + \min_{k_1 : k_1 < k, k_1 \text{is even}} \{T(k_1)\} + \min_{k_2 : k_2 < k, k_2 \text{is odd}} \{T(k_2)\} \Bigg\} \\ = {} & \min\{-1+1+0, 1+0+1\} = 0. \end{aligned}$$

Similarly, if $k$ is odd, then we have

$$\begin{aligned} T(k) = \min \Bigg\{ & -1 + \min_{k_1 : k_1 < k, k_1 \text{is odd}} \{T(k_1)\} + \min_{k_2 : k_2 < k, k_2 \text{is odd}} \{T(k_2)\}, \\ & 1 + \min_{k_1 : k_1 < k, k_1 \text{is even}} \{T(k_1)\} + \min_{k_2 : k_2 < k, k_2 \text{is even}} \{T(k_2)\} \Bigg\} \\ = {} & \min\{-1+1+1, 1+0+0\} = 1. \end{aligned}$$

This proves the induction step and completes the proof of (v) of the theorem. $\square$

## 3 Lower Bounds for $M(3, n)$

Batcher's odd-even merge yields $M(3,n) \leq C(3,n) = \lceil (7n+3)/4 \rceil$. In [1], Aigner and Schwarzkopf proved that,

$$\begin{array}{ll} M(3,n) = C(3,n), & \text{for every } n \equiv 0,1,3 \bmod 4, \textit{and} \\ M(3,n) \geq C(3,n) - 1, & \text{for every } n \equiv 2 \bmod 4. \end{array} \tag{4}$$

In the same paper[1], they also stated that $M(3,n) = C(3,n)$ for $n \equiv 2 \bmod 4$ but without a detailed proof. Recently, Iwata[3] reproved Eq. (4) by showing Eq. (1). In this section, we give a detailed proof for $M(3,n) = C(3,n)$ for every $n \equiv 2 \bmod 4$ based on the results described in the previous section.

**Theorem 2.** *For every $k \geq 0$, $M(3,4k+2) = C(3,4k+2) = 7k+5$.*

Put $n = 4k+2$. Eq. (4) shows $7k+4 \leq M(3,n) \leq C(3,n) = 7k+5$. Throughout this section, suppose for a contradiction that there exists an optimal $(3,4k+2)$-merging network consisting of $7k+4$ comparators with input lines $x_1, x_2, y_1, \ldots, y_{4k+2}, x_3$. First we show that optimal $(2,n)$- and $(1,n)$-merging networks must be "embedded" into the network $N$. To prove this, we shall need some notations and lemmas introduced by Iwata[3].

For a line $z$ and a comparator $c$ that is connected to line $z$, if there is not any comparator positioned to the left of $c$ as far as line $z$ is concerned, then we say that $c$ is the *leftmost* comparator w.r.t. line $z$. If line $z$ is an upper endline (a lower endline) of $c$ then we call $c$ by *downward* leftmost (*upward* leftmost, respectively) w.r.t. line $z$. For example in Fig. 2, the comparator $[y_2 : y_4]$ is the upward leftmost comparator w.r.t. line $y_4$, the comparator $[x_2 : y_2]$ is the downward leftmost comparator w.r.t. line $x_2$ and is also the upward leftmost comparator w.r.t. line $y_2$.

Consider a subnetwork $N_A$ of $N$ consisting of the top three lines $x_1, x_2, y_1$. Obviously, $N_A$ is a $(2,1)$-merging network and $|N_A| \geq 2$. Similarly, a subnetwork $N_B$ consisting of the bottom two lines $y_n, x_3$ is a $(1,1)$-merging network and $|N_B| \geq 1$. Let $N_U$ ($N_D$) be the set of upward (downward, respectively) leftmost comparators w.r.t. line $y_i$ for some $i \in \{2, \ldots, n-1\}$. By the optimality of $N$, $N_U$ and $N_D$ are disjoint. Let $A$ be a network after deletion of the comparators from $N$ that belong to $N_B$ and to $N_D$. Similarly let $B$ be a network constructed from $N$ after deletion of the comparators of $N_A$ and of $N_U$.

The following lemma is proved by Iwata[3].

**Lemma 3 (Lemma 2.2 in [3]).** *The network $A$ acts as a $(2,n)$-merging network, which consists of lines $x_1, x_2, y_1, y_2, \ldots, y_n$ and $|A| \geq M(2,n)$. Similarly, the network $B$ acts as a $(n,1)$-merging network, which consists of lines $y_1, y_2, \ldots, y_n, x_3$ and $|B| \geq M(1,n)$.* □

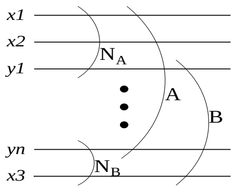


**Fig.4.** $(3,n)$-merging network.

The next lemma says that the networks $A$ and $B$ are in fact optimal. This easily follows from Lemma 3 and the proof is postponed to the appendix due to space constraints.

**Lemma 4.** *Both of the networks $A$ and $B$ is optimal, i.e., $|A| = M(2,n) = 6k+3$ and $|B| = M(1,n) = 4k+2$. Moreover, $|N_A| = 2$, $|N_B| = 1$, $|N_U| = 3k$, $|N_D| = k$ and $|A \cap B| = 3k+1$, and every comparator of $N$ belongs to exactly one of $N_A$, $N_B$, $N_U$, $N_D$ or of $A \cap B$.*

The network $A$ is now shown to be an optimal $(2,n)$-merging network and thus we can apply Theorem 1. Let $A_0$, $A_1$ and $A_{01}$ be subsets of $A$ defined as in Section 2. Since a network consisting of the set of comparators $A_0 \cup A_{01}$ ($B$) acts as an optimal $(1,n)$-merging networks on lines $x_2, y_1, \ldots, y_n$ ($y_1, \ldots, y_n, x_3$, respectively), the following fact is obvious.

**Fact 5.** *For every $1 \leq i \leq n$, the following are true:*

*(i) There is a path from $x_2$ to $y_i$ in $A_0 \cup A_{01}$.*
*(ii) There is a unique comparator in $A_0 \cup A_{01}$ which is of the form $[* : y_i]$.*
*(iii) There is a path from $x_3$ to $y_i$ in $B$.*
*(iv) There is a unique comparator in $B$ which is of the form $[y_i : *]$.* □

From this fact we can easily prove the following:

**Lemma 6.** *For every $1 \leq i \leq n$, there is at most one comparator in $(A_0 \cup A_{01}) \cap B$ that connects to line $y_i$.*

*Proof.* Suppose for a contradiction that there exist two comparators in $(A_0 \cup A_{01}) \cap B$ that connect to line $y_i$ for some $i$. By (ii) and (iv) in Fact 5, one of them is $[y_k : y_i]$ for some $k < i$ and the other is $[y_i : y_j]$ for some $i < j$. Without loss of generality, we can assume that the comparator $[y_k : y_i]$ is left to the comparator $[y_i : y_j]$. Since (iv) of Fact 5, there exist no paths from $x_3$ to $y_k$ in $B$, which contradicts (iii) of Fact 5. □

Now we conclude the proof of Theorem 2.

*Proof (of Theorem 2).* First we claim that $|A_0 \cap B| = 2k$ and $|A_{01} \cap B| = 1$. By similar arguments as in the proof of (iii) of Theorem 1, every comparator in $A_0$ is not leftmost upward, hence $A_0 \cap N_U = \phi$. Since $|A_0| = 2k+1$ ((i) of Theorem 1), $|A_0 \cap N_A| \leq 1$ and $B = N \backslash (N_A \cup N_U)$, we have $|A_0 \cap B| \geq 2k$. It is easy to check that a comparator of the form $[* : y_n]$ in $A_0 \cup A_{01}$ must be in $A_{01}$, and this comparator is also in $B$ since it cannot belong to $N_A$ and to $N_U$. Therefore $|A_{01} \cap B| \geq 1$. On the other hand, Lemma 6 implies $|(A_0 \cup A_{01}) \cap B| \leq n/2 = 2k+1$, and this implies the claim.

For a set of comparator $M$, let $C(M)$ be the set of indices $i$ such that there is a comparator of the form $[* : y_i]$ or of the form $[y_i : *]$ in $M$. By the above claim and Lemma 6, for every $1 \leq i \leq 4k+2$, there is a unique comparator which is of the form $[y_i : *]$ or $[* : y_i]$ in $(A_0 \cup A_{01}) \cap B$, i.e., $C((A_0 \cup A_{01}) \cap B) = \{1, 2, \ldots, 4k+2\}$. On the other hand, since every comparator in $A_0$ has odd length ((v) of Theorem 1), the numbers of even and odd numbers in $C(A_0 \cap B)$ are equal to $2k$. Since every comparator in $A_{01}$ has even length ((iv) of Theorem 1), $C(A_{01} \cap B)$ consists of two even numbers or of two odd numbers. Therefore $C((A_0 \cup A_{01}) \cap B)$ consists of $2k$ even and $2k+2$ odd numbers or of $2k+2$ even and $2k$ odd numbers. This contradicts $C((A_0 \cup A_{01}) \cap B) = \{1, 2, \ldots, 4k+2\}$ since $\{1, 2, \ldots, 4k+2\}$ consists of $2k+1$ even and $2k+1$ odd numbers. This completes the proof of Theorem 2. □

## 4 Lower Bounds for $M(4, n)$

Batcher's $(4, n)$-merging networks use $2n+1$ comparators for $n \equiv 0 \bmod 4$ and $2n+2$ comparators for $n \not\equiv 0 \bmod 4$. Iwata[3] showed that Batcher's $(4, n)$-merging networks are "almost" optimal by proving that $M(4, n) = C(4, n)$ for $n \equiv 0, 1, 3 \bmod 4$ and $M(4, n) \geq C(4, n) - 1$ for $n \equiv 2 \bmod 4$. In this section, we prove that Batcher's $(4, n)$-merging networks are "exactly" optimal for every $n \geq 1$.

**Theorem 7.** *For every $k \geq 0$, $M(4, 4k+2) = C(4, 4k+2) = 8k+6$.*

The outline of the proof is similar to that of the proof for the case $M(3, 4k+2)$: Suppose for a contradiction that there exists an optimal $(4, 4k+2)$-merging network $N$ consisting of $C(4, 4k+2)-1 = 8k+5$ comparators with input lines $x_1, x_2, y_1, \ldots, y_{4k+2}, x_3, x_4$. First we show that two optimal $(2, 4k+2)$-merging networks must be "embedded" into the network $N$, and then we show that this is impossible by the structure of optimal $(2, 4k+2)$-merging networks.

Put $n = 4k+2$. As in the previous section, let $N_A$ ($N_B$) be a subnetwork of $N$ consisting of the top three lines $x_1, x_2, y_1$ (the bottom three lines $y_n, x_3, x_4$, respectively). Let $N_U$ ($N_D$) be the set of leftmost upward (downward, respectively) comparators in $N$ w.r.t. line $y_i$ for some $i \in \{2, \ldots, n-1\}$. Let $A$ ($B$) be a network constructed from $N$ after deletion of the comparators both in $N_B$ and in $N_D$ (both in $N_A$ and in $N_U$, respectively). By similar arguments to the proof of the lower bound on $M(3, n)$, we can prove that both of the networks $A$ and $B$ acts as an optimal $(2, n)$-merging network. The proof of Lemma 8 is similar to that of Lemma 4 and is postponed to the appendix due to space constraints.

**Lemma 8.** *Both of the network $A$ and $B$ behaves like an optimal $(2, 4k+2)$-merging network, i.e., $|A| = |B| = M(2, 4k+2) = 6k+3$. Moreover, $|N_A| = |N_B| = 2$, $|N_U| = |N_D| = 2k$ and $|A \cap B| = 4k+1$, and every comparator of $N$ belongs to exactly one of $N_A$, $N_B$, $N_U$, $N_D$ or of $A \cap B$.* □

Since the networks $A$ and $B$ are shown to be optimal $(2, n)$-merging networks, we can apply Theorem 1 to both networks.

**Lemma 9.** *$N_A$ consists of the two comparators $[x_1 : y_1] \in A_1$ and $[x_2 : y_1] \in A_0$.*

*Proof.* By Lemma 8, $N_A$ behaves like an optimal $(2, 1)$-merging network consisting of lines $x_1, x_2, y_1$. There are two different $(2, 1)$-merging networks that use 2 comparators (Fig. 5). If we assume that $N_A$ is the network shown in Fig. 5 (b), then the comparator $[x_1 : x_2]$ must be in $A_1$ and this contradicts (ii) of Theorem 1. Hence $N_A$ must be the network shown in Fig. 5 (a) and the lemma follows easily. □

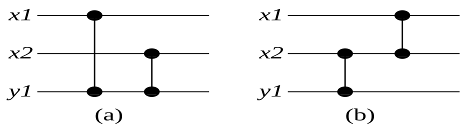


**Fig. 5.** $(2, 1)$-merging networks using 2 comparators.

We divide the set of comparators $A \cap B$ into two subsets $(A \cap B)_E$ and $(A \cap B)_O$ where $(A \cap B)_E$ ($(A \cap B)_O$) consists of all comparators having even length (odd length, respectively).

**Lemma 10.** *For the network $A$, the following are true:*

*(i)* *$A_0$ consists of a comparator $[x_2 : y_1]$ in $N_A$ and $2k$ comparators in $(A \cap B)_O$,*

*(ii)* *$A_{01} \cup A_1$ consists of a comparator $[x_1 : y_1]$ in $N_A$, $2k$ comparators in $N_U$ and $2k+1$ comparators in $(A \cap B)_E$,*

*(iii) For every* $1 \le j \le 4k+2$*, there is a unique comparator in* $A_{01} \cup A_1$ *which is of the form* $[* : y_j]$*.*

*Proof.* By similar arguments used in the proof of (iii) of Theorem 1, every comparator in $A_0$ is not a member of $N_U$, and hence $N_U \subseteq A_{01} \cup A_1$. Since $A = (A \cap B) \cup N_A \cup N_U$, the statements (i) and (ii) of Lemma 10 follow from Theorem 1 and Lemma 9. The statement (iii) is identical to (iii) of Theorem 1. □

By applying all arguments for the network $A$ "upside down", we can obtain a similar result to Lemma 10 for the network $B$. Let $\boldsymbol{b}_0 = \langle 1, 2, \ldots, n, 0, n+1 \rangle$ and $\boldsymbol{b}_1 = \langle 1, 2, \ldots, n, -1, 0 \rangle$. We divide the set of comparators $B$ into three subsets $B_0, B_1$ and $B_{01}$ as follows:

$B_0$ : the set of comparators that change the content of $\boldsymbol{b}_0$, but not of $\boldsymbol{b}_1$,
$B_1$ : the set of comparators that change the content of $\boldsymbol{b}_1$, but not of $\boldsymbol{b}_0$,
$B_{01}$ : the set of comparators that change the contents of $\boldsymbol{b}_0$ and $\boldsymbol{b}_1$.

The next lemma is proved by symmetry.

**Lemma 11.** *For the network* $B$*, the following are true:*

*(i)* $B_0$ *consists of a comparator* $[y_{4k+2} : x_3]$ *in* $N_B$ *and* $2k$ *comparators in* $(A \cap B)_O$*.*
*(ii)* $B_{01} \cup B_1$ *consists of a comparator* $[y_{4k+2} : x_4]$ *in* $N_B$*,* $2k$ *comparators in* $N_D$ *and* $2k+1$ *comparators in* $(A \cap B)_E$*,*
*(iii) For every* $1 \le j \le 4k+2$*, there is a unique comparator in* $B_{01} \cup B_1$ *which is of the form* $[y_j : *]$*.*

Now we conclude the proof of Theorem 7.

*Proof (of Theorem 7).* By Lemmas 10 and 11, we have

$$(A_{01} \cup A_1) \backslash (N_A \cup N_U) = (A \cap B)_E = (B_{01} \cup B_1) \backslash (N_B \cup N_D). \tag{5}$$

For a set of comparators $M$, let $L(M)$ ($U(M)$) be the set of indices $i$ such that there is a comparator of the form $[* : y_i]$ ($[y_i : *]$, respectively) in $M$. By Eq. (5) and Lemma 10, we have $L((A \cap B)_E) = \{2, \ldots, 4k+2\} \backslash L(N_U)$, and by Eq. (5) and Lemma 11, we have $U((A \cap B)_E) = \{1, \ldots, 4k+1\} \backslash U(N_D)$. By the definition of leftmost comparator and Lemma 8, we have $|L(N_U)| = |U(N_D)| = 2k$, $L(N_U) \cap U(N_D) = \emptyset$ and $L(N_U) \cup U(N_D) = \{2, \ldots, 4k+1\}$.

Suppose that $L(N_U)$ contains $d$ even numbers. Then $U(N_D)$ contains $2k-d$ even numbers, $L((A \cap B)_E)$ contains $2k+1-d$ even numbers and $U((A \cap B)_E)$ contains $2k-(2k-d) = d$ even numbers. On the other hand, since every comparator in $(A \cap B)_E$ has even length, the numbers of even numbers in $L((A \cap B)_E)$ and that in $U((A \cap B)_E)$ must be the same. This contradicts the fact that $2k+1-d \neq d$ for every integer $d$. □

## 5 Some More Results and Concluding Remarks

Our theorem on the structure of optimal $(2, n)$-merging networks described in Section 2 can also be used to determine $M(m, n)$ for some $m > 4$. For example, we can show the following theorem. Note that Iwata[3] has shown that $M(5, n) = C(5, n)$ for $n \equiv 0, 1, 5$ mod 8 and $M(m, n) = C(m, n)$ for $m = 6, 7, 8$ and $n \equiv m$ mod 8.

**Theorem 12.** *For every $k \geq 0$, $M(5, 8k+6) = C(5, 8k+6) = 17k+16$.*

*Proof (sketch).* Suppose for a contradiction that there is a $(5, 8k+6)$-merging network $N$ consisting of lines $x_1, x_2, y_1, \ldots, y_{8k+6}, x_3, x_4, x_5$ that uses $17k+15$ comparators. Let $N_A$ ($N_B$) be a subnetwork of $N$ of the top three lines $x_1, x_2, y_1$ (the bottom four lines $y_{8k+6}, x_3, x_4, x_5$. Let $N_U$ ($N_D$) be the set of upward (downward, respectively) leftmost comparators w.r.t line $y_i$ for some $i \in \{2, \ldots, 8k+5\}$. By similar calculations in the proof of Lemma 4, we have $|A| = M(2, 8k+6)$, $|B| = M(3, 8k+6)$, $|N_A| = 2$ and $|N_B| = 3$. This implies that every comparator of $N$ belongs to exactly one of $N_A$, $N_B$, $N_U$, $N_D$ or of $A \cap B$.

Now we consider a subnetwork $N'_B$ of $N$ consisting of the bottom five lines $y_{8k+5}, y_{8k+6}, x_3, x_4, x_5$. Since $N'_B$ forms $(2,3)$-merging network, we have $|N'_B| \geq 5$. This implies that there are at least two comparators in $N'_B$ of the form $[y_{8k+5} : *]$. Since every comparator of $N$ belongs to exactly one of the sets $N_A$, $N_B$, $N_U$, $N_D$ or of $A \cap B$, there must be a comparator $[y_{8k+5}, y_{8k+6}]$ in $A \cap B$. (One possible comparator of the form $[y_{8k+5} : *]$ in $N'_B$ may be in $N_D$ for line $y_{8k+5}$.) By Theorem 1, this must be in $A_0$ and there is another comparator of the form $[* : y_{8k+6}]$ in $A_1$. On the other hand, since $M(2, 8k+6) = M(2, 8k+5)+1$, there is exactly one comparator of the form $[* : y_{8k+6}]$ in $A$, a contradiction. $\square$

Finally, we remark that the arguments described in Section 4 can also be used to characterize the structure of optimal $(4, 4k)$-merging networks, which is stated below. We believe that this would help to determine $M(m, n)$ for higher values of $m$ and $n$.

**Proposition 13.** *Let $N$ be an optimal $(4, 4k)$-merging network consisting of lines $x_1, x_2, y_1, \ldots, y_{4k}, x_3, x_4$. Then the networks $A$ and $B$, defined as in Section 4, are optimal $(2, 4k)$-merging networks. Moreover, Eq. (5) holds and both of $L((A \cap B)_E)$ and $U((A \cap B)_E)$ consists of $k$ even and $k$ odd numbers.* $\square$

## References

1. M. Aigner and O. Schwarzkopf, "Bounds of the Size of Merging Networks", *Discrete Applied Mathematics*, Vol. 61, pp. 187–194, 1995.
2. K.E. Batcher, "Sorting Networks and Their Applications", *Proc. AFIPS 1968 SJCC*, Vol. 32, pp. 307–314, 1968.
3. S. Iwata, "Lower Bounds for Merging Networks", *Information and Computation*, Vol. 168, pp. 187–205, 2001
4. D.E. Knuth, *The Art of Computer Programming Vol. 3: Sorting and Searching, 2nd ed.*, Addison-Wesley, 1998.
5. T. Leighton, Y. Ma and T. Suel, "On Probabilistic Networks for Selection, Merging, and Sorting", *Proc. 7th ACM Symp. Parallel Algorithms and Architectures*, pp. 106–118, 1995.
6. P.B. Miltersen, M. Paterson, and J. Tarui, "The Asymptotic Complexity of Merging Networks", *J. Assoc. Comput. Mach.*, Vol. 43, pp. 147–165, 1996 (A preliminary version appeared in *Proc. 33rd FOCS*).
7. K. Yamazaki, H. Mizuno, K. Masuda and S. Iwata, "Minimum Number of Comparators in (6,6)-merging Network", *IEICE Trans. Inform. Syst.*, Vol. E83-D, pp. 137–141, 2000.
8. A.C. Yao and F.F. Yao, "Lower Bounds of Merging Networks", *J. Assoc. Comput. Mach.*, Vol. 23, pp. 566–571, 1976.

## Appendix

*Proof (of Lemma 4).* By Lemma 3, $|A| \geq M(2,n) = 6k+3$ and $|B| \geq M(1,n) = 4k+2$. Since $A \oplus B \supseteq N_A \cup N_B \cup N_U \cup N_D$, we have

$$\begin{aligned} 14k+8 = 2|N| &\geq |A| + |B| + |A \oplus B| \\ &\geq (6k+3) + (4k+2) + (2+1+4k) = 14k+8. \end{aligned}$$

Thus $|A| = 6k+3$, $|B| = 4k+2$, $|N_A| = 2$ and $|N_B| = 1$. Since $|A| = |N| - |N_B| - |N_D| = (7k+4) - 1 - |N_D|$, $|N_D| = k$ holds. This implies $|N_U| = 3k$.

If we set $N_R = N \backslash (N_A \cup N_B \cup N_D \cup N_U)$, we have $|N_R| = 3k+1$. Since $A$ contains $(A \cap N_R) \cup N_A \cup N_U$, $|A \cap N_R| + |N_A| + |N_U| = |A \cap N_R| + 2 + 3k \geq 6k+3$ holds and this implies $|A \cap N_R| = 3k+1 = |N_R|$. Similarly, $|B \cap N_R| = |N_R|$. This concludes that $A \cap B = N_R$. □

*Proof (of Lemma 8).* The proof is analogous to that of Lemma 4. We use the next lemma which corresponds to Lemma 3.

**Lemma 14 (Lemma 2.2 in [3]).** *The network $A$ acts as a $(2,n)$-merging network, which consists of lines $x_1, x_2, y_1, y_2, \ldots, y_n$ and $|A| \geq M(2,n)$. Similarly, the network $B$ acts as a $(n,2)$-merging network, which consists of lines $y_1, y_2, \ldots, y_n, x_3, x_4$ and $|B| \geq M(2,n)$.* □

By Lemma 14, we have $|A| \geq M(2,4k+2) = 6k+3$ and $|B| \geq M(2,4k+2) = 6k+3$. Since $A \oplus B \supseteq N_A \cup N_B \cup N_D \cup N_U$ holds, we have

$$\begin{aligned} 16k+10 = 2|N| &\geq |A| + |B| + |A \oplus B| \\ &\geq M(2,4k+2) + M(2,4k+2) + (2+2+4k) \geq 16k+10 \end{aligned}$$

Thus $|A| = |B| = 6k+3 = C(2,4k+2)$ and $|N_A| = |N_B| = 2$. Since $6k+3 = |A| = |N| - |N_B| - |N_D| = (8k+5) - 2 - |N_D|$ holds, $|N_D| = 2k$. Similarly, we have $|N_U| = 2k$.

If we set $N_R = N \backslash (N_A \cup N_B \cup N_D \cup N_U)$, we have $|N_R| = 4k+1$. Since $A$ contains $(A \cap N_R) \cup N_A \cup N_U$, $|A \cap N_R| + |N_A| + |N_U| = |A \cap N_R| + 2 + 2k \geq 6k+3$ holds and this implies $|A \cap N_R| = 4k+1 = |N_R|$. Similarly, $|B \cap N_R| = |N_R|$. This concludes that $A \cap B = N_R$ holds.